Solution of DPP # 2

TARGET : JEE (ADVANCED) 2015

Course: VIJETA & VIJAY (ADP & ADR)

MATHEMATICS

1_. $f(f(x)) = \frac{1}{f(x)} \quad \Rightarrow \quad f(y) = \frac{1}{y}$ when y is in the range of f

$f(11) = 10 \quad \Rightarrow \quad$ 10 is in range of f $\quad \Rightarrow \quad f(10) = \frac{1}{10}$

So 10 and $\frac{1}{10}$ are in range of f. So by IVT, 9 is in range of f $\quad \Rightarrow \quad f(9) = \frac{1}{9}$.

2_. $\lim\limits_{n\to\infty}\left(\frac{\sin\frac{2}{n}}{2\tan\frac{1}{n}} + \frac{1}{n^2+\cos n}\right)^{n^2} = \lim\limits_{n\to\infty}\left(\cos^2\frac{1}{n} + \frac{1}{n^2+\cos n}\right)^{n^2} = e^{\lim\limits_{n\to\infty}\left(\frac{n^2}{n^2+\cos n} - n^2\sin^2\frac{1}{n}\right)} = e^{1-1} = e^{\circ} = 1$

3_. $\ell = \frac{1}{3}\left(\ell + \frac{5}{\ell}\right) \quad \Rightarrow \quad 3\ell^2 = \ell^2 + 5 \quad \Rightarrow \quad 2\ell^2 = 5 \quad \Rightarrow \quad \ell^2 = \frac{5}{2}$

4_. $f(2x^2 - 1) = -2x^3 f(-x) \quad \Rightarrow \quad 2x^3 f(x) = -2x^3 f(-x)$

$\Rightarrow \quad f(-x) = -f(x) \quad \Rightarrow \quad f(x)$ is odd $\quad \Rightarrow \quad f^{2010}(x)$ is odd $\quad \Rightarrow \quad f^{2010}(0) = 0$

5_. $f'(0) = \lim\limits_{h\to 0}\frac{f(h)-f(0)}{h} = \lim\limits_{h\to 0}\frac{\frac{hg(h)}{|h|} - 0}{h} = \lim\limits_{h\to 0}\frac{g(h)}{|h|} = \lim\limits_{h\to 0}\frac{g(h)-g(0)}{|h|}$

$\Rightarrow \quad f'(0^+) = \lim\limits_{h\to 0^+}\frac{g(h)-g(0)}{h} = g'(0^+) = 0$ Also $f'(0^-) = \lim\limits_{h\to 0^-}\frac{g(h)-g(0)}{-h} = -g'(0^-) = 0$

6_. $\frac{dy}{dx} = 1 + e^x \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{1+e^x} \quad \Rightarrow \quad \frac{d^2x}{dy^2} = \frac{-e^x}{(1+e^x)^3}$

7_. As $|a-1| + |b-1| = |a| + |b| = |a+1| + |b+1|$ so a & b cannot be of same sign

Let $a \ge 0$ & $b \le 0$ then $|a-1| - (b-1) = a - b \quad \Rightarrow \quad |a-1| = a-1 \quad \Rightarrow a \ge 1$ & $b \le 0$

Similarly $a \le 0$ & $b \ge 1$ so $a \ge 1$ and $b \le 0$ or $a \le 0$ & $b \ge 1$ ...(i)

Also given that $|a| + |b| = |a+1| + |b+1|$

$\Rightarrow \quad a \ge 0$ & $b \le -1$ or $a \le -1$ & $b \ge 0$ ...(ii)

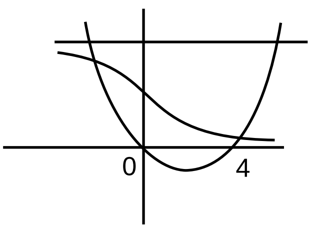
From (i) & (ii) $a \ge 1$ & $b \le -1$ or $a \le -1$ & $b \ge 1$

$\Rightarrow \quad a - b \ge 2$ or $a - b \le -2$

8_. $(x - x^2 - 1)^2 = (2x - 4 - x^2)^2 \quad \Rightarrow \quad x - 3 = 0$ or $-2x^2 + 3x - 5 = 0$

$\Rightarrow \quad x = 3$

9_.



10. $\lim\limits_{x\to\frac{\sqrt3}{2}} \dfrac{\frac{\pi}{3}-\left(\pi-2\sin^{-1}x\right)}{x-\frac{\sqrt3}{2}} = \lim\limits_{x\to\frac{\sqrt3}{2}} \dfrac{2\sin^{-1}x-\frac{2\pi}{3}}{x-\frac{\sqrt3}{2}} = \lim\limits_{x\to\frac{\sqrt3}{2}} \dfrac{2}{\sqrt{1-x^2}} = 4$ (By ' L' Hostpital Rule)

11^. $\lim\limits_{n\to\infty}\left[\frac12+\frac{1}{2^2}+...+\frac{1}{2^n}\right] = \lim\limits_{n\to\infty}\left[\dfrac{\frac12\left(1-\frac{1}{2^n}\right)}{1-\frac12}\right] = \lim\limits_{n\to\infty}\left[1-\frac{1}{2^n}\right] = 0$ as $\frac{1}{2^n}\to 0^+$

12. $f'(0) = \lim\limits_{x\to0}\dfrac{\frac{(e^x-1)(x\sin x)}{x^n}-0}{x} = \lim\limits_{x\to0}\left(\frac{\sin x}{x^{n-1}}\right) = \lim\limits_{x\to0}\left(\frac{1}{x^{n-2}}\right) \Rightarrow n-2\le 0 \Rightarrow n\le 2$

13. $f(x+y) = f(x)\,f(y) \Rightarrow f(x) = ke^{xf'(0)}$ where k is a constant $\Rightarrow f(x) = ke^{2x}$
As $f'(0) = 2 \Rightarrow k = 1 \Rightarrow f(x) = e^{2x}$

14. $f(p) = -f(q) \Rightarrow$ either $f(p)\,f(q) < 0$ or $f(p) = 0 = f(q)$
$\Rightarrow$ exactly one root in (p, q) or roots are p and q

15_. $24-2x-x^2 > 0,\ a > 0,\ a \ne 1$
$\Rightarrow x^2+2x-24 < 0,\ x^2-25 < 0,\ x \ne \pm 3$
$\Rightarrow x \in (-5,-3)\cup(-3,3)\cup(3,4)$

case(i) : $x \in (-5,-3)\cup(3,4) \Rightarrow \dfrac{25-x^2}{16} < 1 \Rightarrow 0 < a < 1$

so given equation becomes $\dfrac{24-2x-x^2}{14} < a \Rightarrow \dfrac{24-2x-x^2}{14} < \dfrac{25-x^2}{16}$

$\Rightarrow x^2+16x-17 > 0 \Rightarrow x \in (-\infty,-17)\cup(1,\infty) \Rightarrow x\in(3,4)$

case(ii) : $x \in (-3,3) \Rightarrow a \in \left(1, \frac{25}{16}\right]$

so given equation becomes $\dfrac{24-2x-x^2}{14} > a \Rightarrow x \in (-17, 1) \Rightarrow x \in (-3, 1)$

from case (i) & (ii) we get $x \in (-3,1)\cup(3,4)$

16^*. $f(x) - x^2 = (x-1)(x-2)(x-3)(x-4)(x-5)$

17_*. $a = \lim\limits_{x\to0^-} f(x) = 0^{\frac{1}{-1}} = \frac{1}{0}$ = undefined ; $b = \lim\limits_{x\to0^+} f(x) = 1^{\frac{1}{0}}$ = undefined

18_*. $\tan^{-1}(x+2) + \cot^{-1}\sqrt{4x+20} = \frac{\pi}{2} \Rightarrow x+2 = \sqrt{4x+20} \Rightarrow x = 4 \Rightarrow a = 4$

$\Rightarrow \lim\limits_{x\to0}\dfrac{\sqrt[4]{1+x}-\frac{(1+x)^{\frac1x}}{e}-bx}{5x+kx^2+x^3} = 0 \Rightarrow \lim\limits_{x\to0}\dfrac{\left(\frac34-b\right)x-\frac{53}{96}x^2+....}{5x+kx^2+x^3} = 0 \Rightarrow \frac34 - b = 0,\ k\in R$

19_*. $f(0) = 0 = \lim\limits_{x\to0^+} f(x) \Rightarrow \lim\limits_{x\to0^-} f(x) = \lim\limits_{x\to0^-}\dfrac{[\tan x]}{[x]} = \dfrac{-1}{-1} = 1$

20_*. Continuous $\Rightarrow 0 = \sin^{-1}b \Rightarrow b = 0$
Differentiable $\Rightarrow f'(0^-) = f'(0^+) \Rightarrow 0 = 1 \Rightarrow$ not possible

21_*. Let $\lim\limits_{x\to\infty} f(x) = \lambda(\neq 0)$

For any a > 0, $\lambda = \lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} f(ax) = \lim\limits_{x\to\infty} f(a)\, f(x) = f(a) \lim\limits_{x\to\infty} f(x) = \lambda f(a)$

$\Rightarrow \quad f(a) = 1 \quad \Rightarrow \quad f(1) = 1,\ f(2) = 1......$

Also $f(0) = \lim\limits_{x\to 0^+} f(x) = 1$.

22_*. Around x = 2, $f(x) = |\sin(x-1) - 2| = 2 - \sin(x-1) \quad \Rightarrow \quad f'(x) = -\cos(x-1) \Rightarrow \quad f'(2) = -\cos 1$

Now $f(x) = \begin{cases} 2-\sin(x+1) & , \ x \to 0^- \\ 2+\sin(x-1) & , \ x \to 0^+ \end{cases} \quad \Rightarrow \quad f'(0^-) = -\cos 1,\ f'(0^+) = \cos 1$

23_*. $\alpha + \dfrac{1}{\alpha} > 2 \quad \Rightarrow \quad \dfrac{-b}{a} > 2 \quad \Rightarrow \quad \dfrac{2a+b}{a} < 0$

$\Rightarrow \quad a(2a + b) < 0 \quad \Rightarrow \quad af'(1) < 0$

24_*. $(\lambda - 2)(x^2 + x + 1) - (\lambda + 2)(x^2 - x + 1) = 0 \quad \Rightarrow \quad 4x^2 - 2\lambda x + 4 = 0$

$D = 0 \quad \Rightarrow \quad \lambda = \pm 4$

25_*. $D \geq 0 \ \ \forall \lambda \in R \quad \Rightarrow \quad (b - \lambda)^2 - 4a(a - b - \lambda) \geq 0$

$\Rightarrow \quad \lambda^2 + 2(2a - b)\lambda + b^2 + 4ab - 4a^2 \geq 0$

$D \leq 0 \quad \Rightarrow \quad a(a - b) \leq 0$

26*^. $|x^2 - 2x| + |4 - x| > |x^2 - 3x + 4| \quad \Rightarrow \quad (x^2 - 2x)(4 - x) \not\geq 0 \quad \Rightarrow \quad (x^2 - 2x)(4 - x) < 0$

$\Rightarrow \quad x \in (0, 2) \cup (4, \infty)$

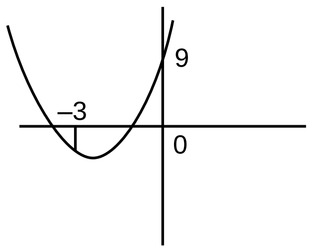
27^*. $$\lim_{n\to\infty} \left\{ \frac{(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{4}\right)....\left(n+\frac{1}{2^{n-1}}\right)}{n^n} \right\}^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{2n}\right)^n \left(1+\frac{1}{4n}\right)^n \ .\ .\ .\ .\ . \left(1+\frac{1}{2^{n-1}.\ n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1+\frac{1}{2n}\right)^{\frac{2n}{2}} \left(1+\frac{1}{4n}\right)^{\frac{4n}{4}} \ .\ .\ .\ .\ .\ .\ . \left(1+\frac{1}{2^{n-1}\ .\ n}\right)^{\frac{2^{n-1}\ .\ n}{2^{n-1}}} = \lim_{n\to\infty} \left(e^1 . e^{\frac{1}{2}} . e^{\frac{1}{4}} .... e^{\frac{1}{2^{n-1}}} \right)$$

$$= e^{1 + \frac{1}{2} + \frac{1}{4} + . . . \infty} = e^2$$

28^*. Possible points of discontinuity of f(x) are 4x = 0, 1, 2, . . , 20 and 3x = 1, 2, 4, 5, 7, 10, 11, 13, 14 so total 30 points but f(x) is continuous at x = 0, 1, 2, 3, 4, 5

29^*.



Case-i $x > 0, \alpha > 0 \quad \Rightarrow \quad (x + 3)^2 = \alpha x$

$\Rightarrow \quad x^2 + (6 - \alpha)x + 9 = 0 \quad \Rightarrow \quad D = 0 \quad \Rightarrow \quad \alpha = 12$

Case-ii $-3 < x < 0, \alpha < 0 \quad \Rightarrow \quad x^2 + (6 - \alpha)x + 9 = 0$

$f(-3) < 0 \quad \Rightarrow \quad 9 - 18 + 3\alpha + 9 < 0 \quad \Rightarrow \quad \alpha < 0$

30^*. 1 and 2 lie between the roots $\Rightarrow$ f(1) < 0 and f(2) < 0

$\Rightarrow \quad 3a + 1 < 0$ and $4 + 5a < 0 \quad \Rightarrow \quad a < -\dfrac{4}{5}$

31*. $(1-h)^{\infty} = 0$, $1^{\infty} = 1$ and $(1+h)^{\infty} = \infty$ where $h \to 0^{+}$

32*. $f(x) = \lim\limits_{x\to\infty} x \sin b\, \ell n\left(1+\dfrac{1}{x\sin a}\right) = \lim\limits_{x\to\infty} \dfrac{x\sin b}{x\sin a} = \dfrac{\sin b}{\sin a} \in (0, \infty)$

33_. f(x) is not defined at $x = \pm 1$

$\dfrac{\sin|x|}{1-|x|^2}$ is continuous at x = 0, so $\dfrac{x\sin|x|}{1-|x|^2}$ is differentiable at x = 0

34_. $f(x) = (x-3)|(x-3)(x-4)| + \cos(x-3)$

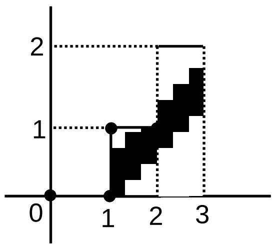
$|(x-3)(x-4)|$ is continuous at x = 3 so $(x-3)|(x-3)(x-4)|$ is differentiable at x = 3

But f(x) is non-differentiable at x = 4 as $f'(4^-) = -1$ and $f'(4^+) = 1$

35_. $h(x) = \sin x|\sin x| \Rightarrow$ $h(x) = \begin{cases} -\sin^2 x, & -\pi \le x \le 0 \\ \sin^2 x, & 0 \le x \le \pi \end{cases}$ $\Rightarrow$ h(x) is differentiable at x = 0

Also h(x) is periodic with period 2π so differentiable everywhere.

36. $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ x-1, & 1 \le x < 2 \\ 2x-3, & 2 \le x < 3 \end{cases}$



37. f(0) not defined so discontinuous at x = 0

38. $\phi(x) - \phi\left(\dfrac{x}{2}\right) = \phi\left(\dfrac{x}{2}\right) - \phi\left(\dfrac{x}{4}\right) + x^2$; $\phi\left(\dfrac{x}{2}\right) - \phi\left(\dfrac{x}{4}\right) = \phi\left(\dfrac{x}{4}\right) - \phi\left(\dfrac{x}{8}\right) + \dfrac{x^2}{4}$

$\phi\left(\dfrac{x}{4}\right) - \phi\left(\dfrac{x}{8}\right) = \phi\left(\dfrac{x}{8}\right) - \phi\left(\dfrac{x}{16}\right) + \dfrac{x^2}{16}$

. . . .

$\phi\left(\dfrac{x}{2^{n-1}}\right) - \phi\left(\dfrac{x}{2^n}\right) = \phi\left(\dfrac{x}{2^n}\right) - \phi\left(\dfrac{x}{2^{n+1}}\right) + \dfrac{x^2}{\left(2^{n-1}\right)^2}$

Adding all, we get

$\phi(x) - \phi\left(\dfrac{x}{2^n}\right) = \phi\left(\dfrac{x}{2}\right) - \phi\left(\dfrac{x}{2^{n+1}}\right) + \dfrac{4}{3}x^2$

Take limits as $n \to \infty$

$\Rightarrow$ $\phi(x) - \phi(0) = \phi\left(\dfrac{x}{2}\right) - \phi(0) + \dfrac{4}{3}x^2$ $\Rightarrow$ $\phi(x) - \phi\left(\dfrac{x}{2}\right) = \dfrac{4}{3}x^2$

Repeating the same process $\Rightarrow$ $\phi(x) - \phi\left(\dfrac{x}{2}\right) = \dfrac{4}{3}x^2$

Repeating the same process, $\Rightarrow$ $\phi(x) - \phi(0) = \dfrac{16}{9}x^2$ $\Rightarrow$ $y - 1 = \dfrac{16}{9}x^2$

39. vertex is (0, 1)

40. Length of latus rectum is $\dfrac{9}{16}$.